State Space Design: Controllability & Observability

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

State space techniques emerged around 1960. They are direct and exploit the efficient computations of linear algebra.

- State space models, modes & similarity transformations
- Controllability & Observability
- Special forms of state equations
- State Space to Transfer Function

Similarity Transformations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$x \in R^n, u \in R^m, y \in R^p$

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Now consider the transformation to new states z, defined by

$$x = Tz \Leftrightarrow z = T^{-1}x$$

$$T\dot{z} = ATz + Bu$$
$$y = CTz + Du$$
$$\Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du$$

so that,

$$\dot{z} = A^*z + B^*u$$
$$y = C^*z + D^*u,$$

$$A^* = T^{-1}AT, \; B^* = T^{-1}B, \; C^* = CT, \; D^* = D$$

Diagonal Form

$$\begin{array}{lllll} \text{eigen-system} & & & & \\ \text{of } A: & \lambda_1 & \lambda_2 & \cdots & \lambda_n & \leftarrow \text{eigenvalues} \\ & h_1 & h_2 & \cdots & h_n & \leftarrow \text{independent eigenvectors} \end{array}$$

$$T \triangleq [h_1 \quad h_2 \quad \cdots \quad h_n]$$

$$\Rightarrow A^* = [h_1 \quad h_2 \quad \cdots \quad h_n]^{-1} A [h_1 \quad h_2 \quad \cdots \quad h_n]$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\boxed{\dot{z}_i = \lambda_i z_i + b_i^* u, \ \ i = 1, \dots, n}$$

A decoupled system of n 1st order ode's

Example

Define A

Compute eigensystem

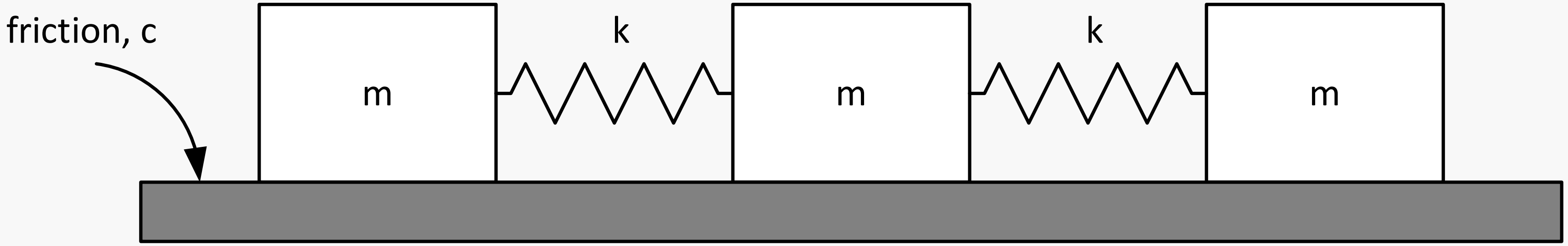
Check similarity trans

Use linear solve rather than inv

```
>> A=[3 2 1;4 5 6;1 2 3];
>> [V,D]=eig(A)
V =
   -0.3482   -0.8581    0.4082
   -0.8704    0.1907   -0.8165
   -0.3482    0.4767    0.4082
D =

    9.0000         0         0
         0    2.0000         0
         0         0   -0.0000
>> inv(V)*A*V
ans =
    9.0000   -0.0000   -0.0000
   -0.0000    2.0000   -0.0000
   -0.0000   -0.0000   -0.0000
>> V\A*V
ans =
    9.0000   -0.0000    0.0000
   -0.0000    2.0000         0
   -0.0000   -0.0000    0.0000
```

3 Mass (1)



$$m\ddot{x}_1 = k\left(x_2 - x_1\right) - c\dot{x}_1$$
$$m\ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) - c\dot{x}_2$$
$$m\ddot{x}_3 = -k\left(x_3 - x_2\right) - c\dot{x}_3$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m \\ -k & k & 0 & c/m & 0 & 0 \\ k & -2k & k & 0 & c/m & 0 \\ 0 & k & -k & 0 & 0 & c/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

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3 Mass (2)

```
» A=[0 0 0 1 0 0;0 0 0 0 1 0;0 0 0 0 0 1;-1 1 0 -1 0 0;1 -2 1 0 -1 0;0 1 -1 0 0 -1];

» [v,e]=eig(A)
v =
  Columns 1 through 4
   0.1954 - 0.0589i   0.1954 + 0.0589i   0.4330 - 0.2500i   0.4330 + 0.2500i
  -0.3909 + 0.1179i  -0.3909 - 0.1179i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
   0.1954 - 0.0589i   0.1954 + 0.0589i  -0.4330 + 0.2500i  -0.4330 - 0.2500i
   0.0000 + 0.3536i   0.0000 - 0.3536i  -0.0000 + 0.5000i  -0.0000 - 0.5000i
  -0.0000 - 0.7071i  -0.0000 + 0.7071i   0.0000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.3536i   0.0000 - 0.3536i   0.0000 - 0.5000i   0.0000 + 0.5000i
  Columns 5 through 6
  -0.5774            -0.4082
  -0.5774            -0.4082
  -0.5774            -0.4082
  -0.0000             0.4082
   0.0000             0.4082
  -0.0000             0.4082
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

2 complex conjugate modes (2 eigenvalues/eigenvectors each)

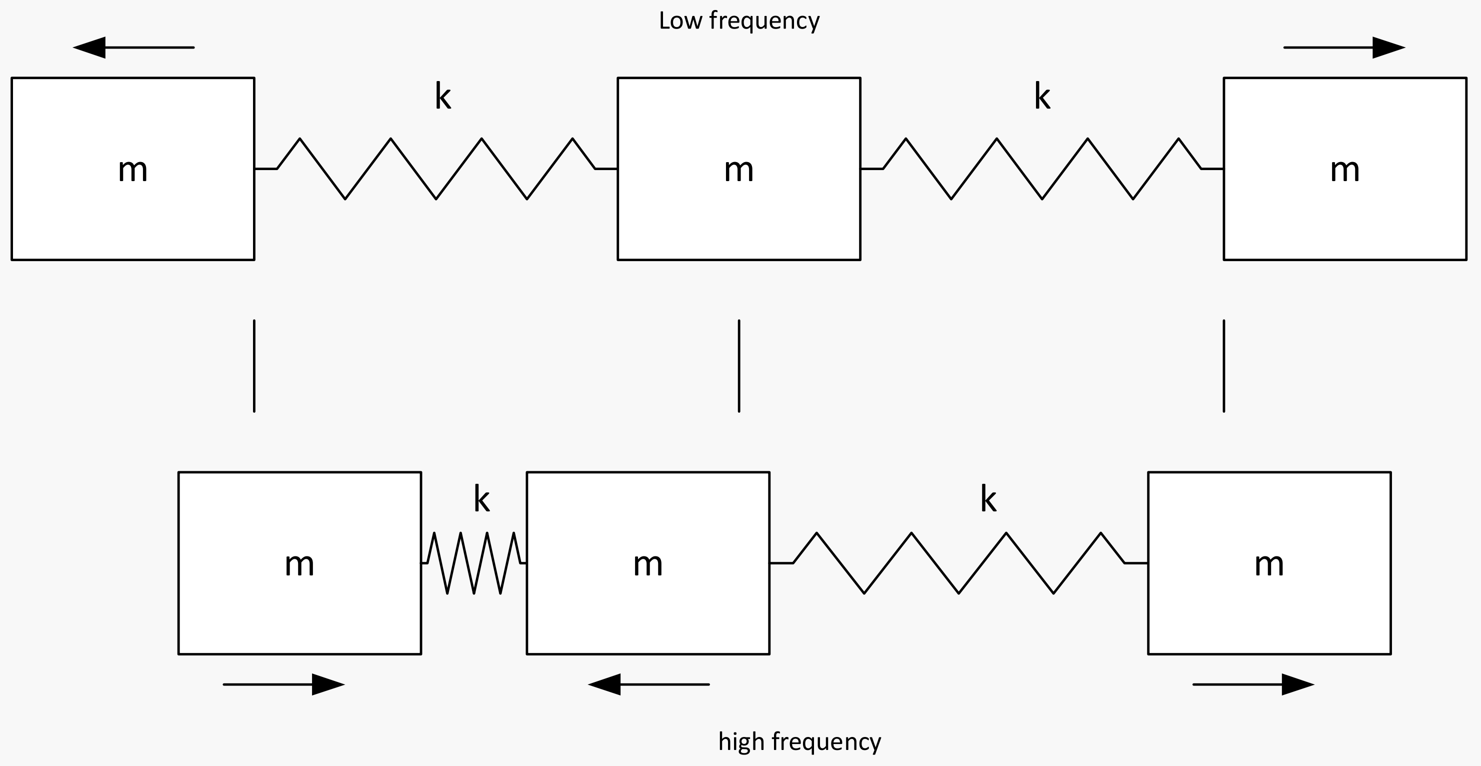
2 real modes (1 eigenvalue/eigenvector each)

3 Mass (3)

```
e =
  Columns 1 through 4
  -0.5000 + 1.6583i        0                   0                   0
        0             -0.5000 - 1.6583i        0                   0
        0                  0              -0.5000 + 0.8660i        0
        0                  0                   0              -0.5000 - 0.8660i
        0                  0                   0                   0
        0                  0                   0                   0
  Columns 5 through 6
        0                  0
        0                  0
        0                  0
        0                  0
   0.0000                  0
        0             -1.0000
```

3 Mass (4)

Translation
Slow exponential decay
Slow damped oscillation
Fast damped oscillation



Controllability & Observability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Controllability: The system is (completely) controllable if there exists a control input $u(t)$ defined on a finite time interval $[0, T]$ that steers the system from any initial state x_0 to any final state x_1.

Observability: The system is (completely) observable if the intial state x_0 can be determined from knowledge of the input $u(t)$ and the measurement of the output $y(t)$ over a finite time interval $[0, T]$.

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Controllability / Observability Tests

Controllability Matrix: $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

Observability Matrix: $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$$\text{Controllable} \Leftrightarrow \text{Rank}\, \mathcal{C} = n$$

$$\text{Observable} \Leftrightarrow \text{Rank}\, \mathcal{O} = n$$

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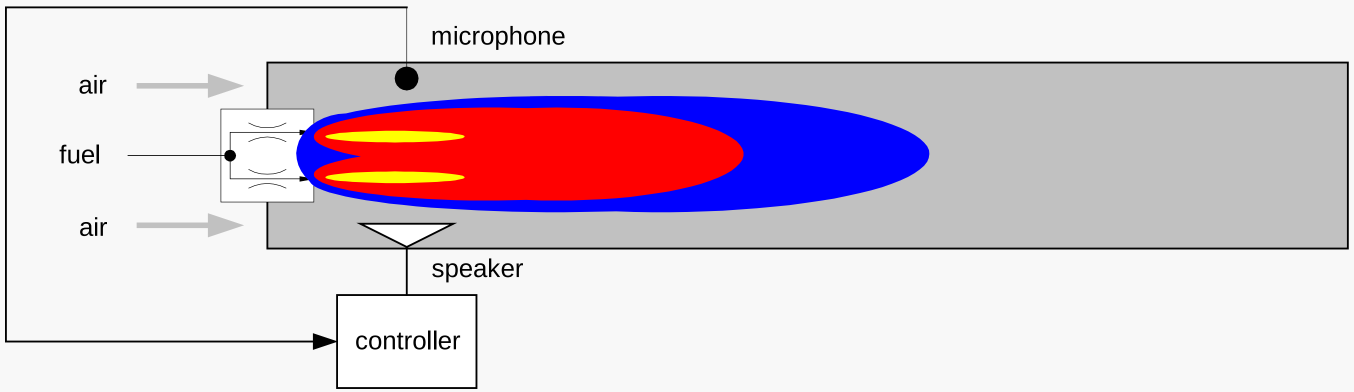
Combustion Example



$$G(s) = K\underbrace{\left(\frac{s+z_f}{s+p_f}\right)}_{\text{flame}}\underbrace{\frac{\left(s^2+2\rho_z\omega_z s+\omega_z^2\right)}{\left(s^2-2\rho_1\omega_1 s+\omega_1^2\right)\left(s^2+2\rho_2\omega_2 s+\omega_2^2\right)}}_{\text{acoustics}}$$

Combustion Example

```
>> s=tf('s');
zf = 1.500; pf = 1.000; rhoz = -0.045; omegaz = 4.500;
rho1 = 0.5; omega1 = 1.0; rho2 = 0.3; omega2 = 3.500;
Gp=((s + zf)*(s^2 + 2*rhoz*omegaz*s + omegaz^2))/((s + pf)*(s^2 -
2*rho1*omega1*s + omega1^2)*(s^2 + 2*rho2*omega2*s + omega2^2));
Gpss=ss(Gp,'min');
[A,B,C,D]=ssdata(Gpss)
A =
   -2.1000   -1.5313   -0.0313   -0.0164   -0.0479
    8.0000         0         0         0         0
         0    4.0000         0         0         0
         0         0    4.0000         0         0
         0         0         0    2.0000         0
B =
    0.5000
         0
         0
         0
         0
C =
         0    0.2500    0.0684    0.3069    0.2373
D =
     0
```

Combustion Example Cont'd

```
>> Co=ctrb(A,B)
Co =
    0.5000   -1.0500   -3.9200   20.5945    4.7715
         0    4.0000   -8.4000  -31.3600  164.7560
         0         0   16.0000  -33.6000 -125.4400
         0         0         0   64.0000 -134.4000
         0         0         0         0  128.0000
>> rank(Co)
ans =
     5
>> Ob=obsv(A,C)
Ob =
         0    0.2500    0.0684    0.3069    0.2373
    2.0000    0.2737    1.2277    0.4746         0
   -2.0100    1.8481    1.8359   -0.0328   -0.0957
   19.0060   10.4216   -0.0684   -0.1584    0.0962
   43.4599  -29.3767   -1.2277   -0.1195   -0.9095
>> rank(Ob)
ans =
     5
```

Special Forms

Consider a SISO controllable & observable system

$$\mathscr{C} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}, \det \mathscr{C} \neq 0 \quad \mathscr{O} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}, \det \mathscr{O} \neq 0$$

$$\mathscr{C}^{-1} \triangleq \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad \mathscr{O}^{-1} \triangleq \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}$$

We will consider four state transformations defined by

$$T_1 \triangleq \mathscr{C}, T_2 \triangleq \mathscr{O}, T_3 \triangleq \begin{bmatrix} q_n \\ q_n A \\ \vdots \\ q_n A^{n-1} \end{bmatrix}, T_4 \triangleq \begin{bmatrix} p_n & Ap_n & \cdots & A^{n-1}p_n \end{bmatrix}$$

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Controllability Form for SISO Systems

$$\dot{x} = Ax + bu,\ y = cx$$

$$\mathcal{C} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}, \det \mathcal{C} \neq 0$$

$$T = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \qquad T = \begin{bmatrix} A^{n-1}b & \cdots & Ab & b \end{bmatrix}$$

$$\Downarrow \qquad \Downarrow$$

$$\dot{z} = \begin{bmatrix} 0 & & 0 & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \qquad \dot{z} = \begin{bmatrix} -a_{n-1} & 1 & & 0 \\ \vdots & 0 & \ddots & \\ -a_1 & & \ddots & 1 \\ -a_0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = cTz$$

Note special structure of A, b

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Controllability Form – Proof

$$\dot{z} = (T^{-1}AT)z + (T^{-1}b)u$$

$$T^{-1}T = I \Rightarrow [T^{-1}b \quad T^{-1}Ab \quad \cdots \quad T^{-1}A^{n-1}b] = I$$

$$T^{-1}AT = [T^{-1}Ab \quad T^{-1}A^2b \quad \cdots \quad T^{-1}A^nb]$$

$$T^{-1}b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & Y_1 \\ 1 & & Y_2 \\ & 0 & \vdots \\ 0 & 1 & Y_n \end{bmatrix}, \quad Y = T^{-1}A^nb$$

suppose $\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$,

C–H Thm $\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_0 I = 0$

$$Y = T^{-1}A^nb = -a_{n-1}T^{-1}A^{n-1}b - \cdots - a_0T^{-1}b = \begin{bmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{n-1} \end{bmatrix}$$

$$T^{-1}b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, T^{-1}Ab = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, T^{-1}A^{n-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Observability Form

$$\dot{x} = Ax + bu,\ y = cx$$

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix},\ \det\mathcal{O} \neq 0$$

$$T = \mathcal{O} \Rightarrow$$

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_{n-1} & \cdots & -a_1 & -a_0 \end{bmatrix} z + \left(T^{-1}b\right)u,\ y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z$$

Note special structure of A, c

Observability Form – Proof

$$y = cx$$
$$\dot{y} = cAx,\ \ cb = 0$$
$$\ddot{y} = cA^2x,\ \ cAb = 0$$
$$\vdots$$
$$y^{(n-1)} = cA^{n-1}x,\ \ cA^{n-2}b = 0$$
$$y^{(n)} = cA^n x + u,\ \ cA^{n-1}b = 1$$

$$\begin{aligned} z_1 &= cx \\ z_2 &= cAx \\ &\vdots \\ z_n &= cA^{n-1}x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= cA^nS^{-1}z + u \end{aligned}$$

$$z = Sx$$

Observability Form – Proof, cont'd

$$S = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}, SS^{-1} = I \Rightarrow \begin{bmatrix} cS^{-1} \\ cAS^{-1} \\ \vdots \\ cA^{n-1}S^{-1} \end{bmatrix} = I$$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

$$cA^n S^{-1} = -a_{n-1}cA^{n-1}S^{-1} - \cdots - a_1 cAS^{-1} - a_0 cS^{-1}$$

$$= \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \end{bmatrix}$$

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Summary of SISO Companion Forms

T_1 (Controllability):

$$\dot{z} = \begin{bmatrix} 0 & & 0 & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \bar{c}z$$

T_2 (Observability):

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_{n-1} & \cdots & -a_1 & -a_0 \end{bmatrix} z + \bar{b}u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z$$

T_3 (Controller/phase variable):

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \bar{c}z$$

T_4 (Observer):

$$\dot{z} = \begin{bmatrix} -a_{n-1} & 1 & & 0 \\ \vdots & 0 & \ddots & \\ -a_1 & & \ddots & 1 \\ -a_0 & 0 & & 0 \end{bmatrix} z + \bar{b}u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} z$$

Transfer Functions from State Space

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

take Laplace transform:

$$sX(s) = AX(s) + BU(s) \Rightarrow [sI - A]X(s) = BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$\Downarrow$$

$$Y(s) = \left\{C[sI - A]^{-1}B + D\right\}U(s)$$

$$\Downarrow$$

$$G(s) = C[sI - A]^{-1}B + D$$

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Example

$$\dot{x} = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} \Rightarrow \text{not controllable, } \mathscr{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow \text{observable}$$

$$G(s) = C[sI - A]^{-1} B = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+4 & -2 \\ 0 & s+2 \end{bmatrix}}{(s+2)(s+4)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{(s+2)}{(s+2)(s+4)} = \frac{1}{s+4}$$